Notes on Hamilton's Principle

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October 2, 2000

1 Introduction

In class, we mentioned that the justification for Lagrangian mechanics is something called Hamilton's Principle. Usually, Hamilton's principle is known as the "principle of least action," and it's assumed to say the following: *The path followed by a system in transitioning from a point* $(\vec{x_1}, t_1)$ *to a point* $(\vec{x_2}, t_2)$ *is the path which minimizes the action,* $S[x(t)] = \int_{t_1}^{t_2} \mathcal{L} dt$. In these notes, I will briefly explain why this is incorrect, what the correct statement is, and why (physically) the world works that way. In addition, I'll explain how we know the second end-point of the trajectory – that is, how the system decides "*where* to end up" so that we can calculate the path it takes to get there.

2 Hamilton's Principle

Historically speaking, Hamilton formulated his eponymous principle *after* Lagrangian mechanics had already been conceived of. It is possible to derive Lagrangian mechanics directly from Newtonian mechanics, but formulated in that way it contains no new insights on how the universe works. Hamilton noticed that the Lagrangian formulation could also be derived by assuming that that all physical paths are minimum-action paths, and therefore proposed the axiomatic principle of least action (that is, it's axiomatic because it can't be proved, and must be assumed).

You may recall that the way Lagrangian mechanics works is by extremizing the action along a path by setting a sort of derivative $\frac{\delta S}{\delta \epsilon}$ (the derivative of the action with respect to some parameter of the path) to zero, in analogy with the technique for finding the minimum of a function in calculus. We mentioned that this technique, as in calculus, only guarantees that we have found *one* of three possibilities: a minimum, a maximum, or a point of inflection (if you don't remember what a point of inflection is, then consider that $f(x) = x^3$ has one at $x = 0$; it's neither a min nor a max, but $\frac{df}{dx} = 0$.) Generally, it's assumed that if we find a path where $\frac{\delta S}{\delta \epsilon} = 0$, then according to Hamilton's principle it's a minimum.

In reality, this is not true. Frequently, the path found by applying the Euler-Lagrange equations to the action of a physical system is either a point of inflection or a local maximum! This immediately begs the question: why does the method work? The answer is simple; Hamilton's principle as stated previously is too restrictive. It should be restated as the *principle of stationary action*, saying that the path that a system follows is one where the action is *stationary* with respect to variation in the path – that is, precisely that $\frac{\delta S}{\delta \epsilon} = 0$! Thus, the Lagrangian method always works.

Why is this the case? The principle of least action is, at least, intuitively appealing – it's elegant and satisfying that nature should choose to minimize some quantity, even if we don't know why. This idea of stationary action is much less satisfying – it seems almost unfairly simple that just when it turns out that our method for finding minima is flawed because it also finds maxima and points of inflection, it also turns out that nature will accept either of the others as well! The answer comes from quantum mechanics.

3 A Brief Justification for Stationary Action

Since this is not a class in quantum mechanics, all I'll attempt to do is give a brief rationale for why the paths with stationary action are the ones that the system follows. There are several perspectives on quantum mechanics, each of which yields the same results. This is analogous to the way in which we can derive the same results for a classical system using Newton's laws, Lagrangian techniques, or Hamilton's equations of motion.

One of these perspectives is called the *path-integral approach.* Using this approach, we can calculate the *probability* for a system, in the state corresponding to $x = x_1$ at time $t = t_1$, to be in the state $x = x_2$ at time $t = t_2$. The way we do this is as follows:

* Form a set $\mathcal{P}$ containing every conceivable path $x(t)$ that connects (x_1, t_1) to (x_2, t_2).
  For every path $x(t) \in \mathcal{P}$, calculate the action $S[x(t)]$. Notice that we've mapped each path (a function) to an action (a real number).

* Define the *amplitude functional* $A[x(t)]$ to be the complex exponential of the action: $A[x(t)] = e^{i\hbar^{-1}S[x(t)]}$.

* Now, add up $A[x(t)]$ for *every* $x(t) \in \mathcal{P}$, to get $A_{total}(x_2, t_2; x_1, t_1) = \int_{\text{all paths } x(t)} e^{i\hbar^{-1}S[x(t)]}$.

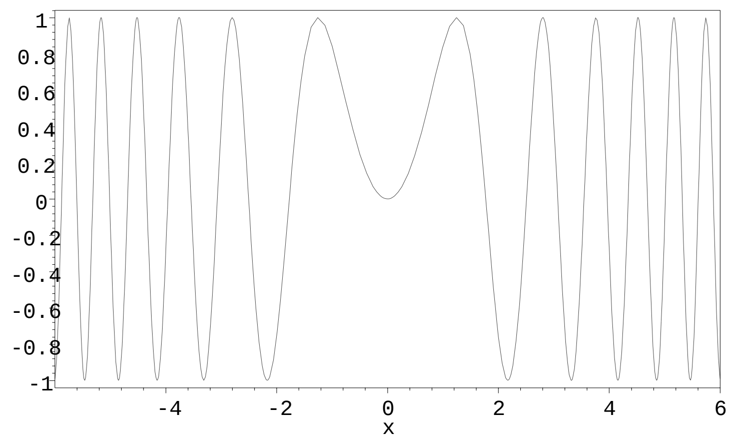
* Finally, the probability for the system to move from x_1 to x_2 is just $|A_{total}|^2$, the norm-squared of the amplitude.

Now, how does this esoteric procedure yield the principle of stationary action? Well, consider the behavior of the function $f(S) = e^{iS}$. As S increases, $f(S)$ oscillates around the unit circle in the complex plane. If you're not familiar with complex functions, just think of $f(S) = sin(S)$; it's pretty much the same for our purposes, except that since $sin(S)$ is a real function, it oscillates up and down in the real numbers. Now, what happens when we integrate this oscillating function of the action over all paths?

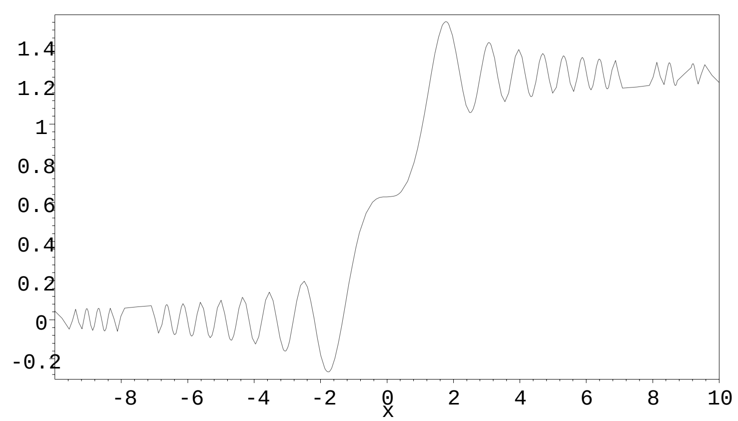
Well, the graphs on the next page should illustrate it. Basically, wherever the action is changing, the integrand e^{iS} is oscillating, and the integral over that region practically vanishes. In the example on the next page, I'm parametrizing my "paths" by a variable called y (not to be confused with the $x(t)$ we were talking about before; this is more like ϵ, but it's easier to use y in Maple, where these plots were done), and the "action" is $S = y^2$. I've shown a graph of the imaginary part of the amplitude, $\Im e^{iS(y)} = sin(S(y))$, and a graph of its integral over an interval from $-\infty tox$. Notice that the integral pretty much just wiggles around without going anywhere *until* we get to around $x = 0$. Why $x = 0$?

Well, at $x = 0$, the derivative of S with respect to the parameter y is zero – which means that the action doesn't depend on the path locally. That is – the action is *stationary* at $y = 0$! Because the integrand isn't oscillating around any more, this path contributes the lion's share of the final integral.

What do we conclude? That the particle has a significant probability of going from x_1 to x_2 *if and only if* there exists at least one path (connecting x_1 to x_2) around which the action is stationary. It doesn't have to be minimal, it doesn't have to be extremal, merely stationary. In classical mechanics, we take certain limits ($\hbar \to 0$, for those of you who like quantum mechanics) so that the phrase "significant probability of" can be replaced by "*any* possibility of". That is, likelihoods become certainties. This is where we get the principle of stationary action from.



$$f(x) = \Im(e^{(i x^2)})$$



$$\int_{-\infty}^{x} \sin(y^2)\, dy$$

4 How the particle knows where to go

Finally, I want to address a more practical question. When we did the calculus of variations to find the minimum action path, we always assumed fixed endpoints x_1 and x_2. How do we know what endpoints to choose? That is, for a given physical system starting at point (x_1, t_1), how do we know at which x_2 it's going to be at time t_2 so as to find the stationary-action path between the two?

The answer is twofold. First of all, a particle will not move under any circumstances between two points that are not connected by a stationary-action path. That is, if by some chance there does not exist any path of minimum or stationary action connecting x_1 to x_2, then the particle will never end up at x_2. This rarely happens, but it is possible – a simple example is a system which contains a region of infinite potential (positive or negative). The system will never end up in that region, because every path connecting the starting point to a point x_2 inside a region of infinite potential has infinite action – which means that there is no minimal or maximal action path. In practice, this is rarely an issue.

Secondly (and more relevantly), there is a sense in which x_2 is *not* determined. That is, there is almost always a path of stationary action connecting any two points x_1 and x_2. However, we haven't specified the initial *velocity* yet! You must remember that all this path-action stuff is dealing with two points in *configuration space*; we aren't talking about phase space. The end points x_1, x_2 determine the path, but that path determines the velocity $\dot{x}(t)$. So, for instance, I could analyze the physics of a ball dropped in a gravitational field, and find a stationary-action path connecting the points $z(t_1) = 0$, $z(t_2) = 0$ – which would imply that the ball does not fall, but stays in place! However, in doing so I find that the path which connects those two points requires the initial velocity of the ball to be upward, which makes sense.

If, however, I insist that the intial velocity of the ball be zero – it's dropped from rest – then the only path going *anywhere* that I can find which is stationary-action is our old friend, $z(t) = -\frac{1}{2}gt^2$. Thus, fixing the initial position *and* velocity (or momentum) is equivalent to fixing the end points.

This leaves at least one question to which I am not sure I have an answer, so you may want to think about it. Why is it, if we are going to claim that knowing $x_{initial}$ and $\dot{x}_{initial}$ is equivalent to $x_{initial}$ and x_{final}, that the conversion from one set of information (initial position-velocity) to the other (initial-final position) is always defined and invertible? That is, why is it that I never fail to find an $[x_1, x_2]$ pair to satisfy a particular $[x_1, \dot{x}_1]$ pair? I don't know yet... maybe you can figure it out.

Cheers,
Robin